International Journal of Theoretical Physics, Vol. 41, No. 8, August 2002 (© 2002)

Trapped Fermions in Gravitational Field

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Received February 4, 2002

Trapped noninteracting Fermi gas in an external gravitational field in Newtonian approximation is considered. Analytical equations for chemical potential, internal energy, and specific heat of trapped Fermi gas are computed. The spatial distribution of completely degenerate fermions in nonhomogeneous gravitational field is calculated. The effects of the influence of gravitational field on Fermi gas are discussed.

KEY WORDS: trapped fermions; gravitational field.

1. INTRODUCTION

The impressive achievements in the experiments with the trapping and cooling of the ensembles of alkali atoms (Anderson *et al.*, 1995; Bradley *et al.*, 1995; Davis *et al.*, 1995; Fried *et al.*, 1998) have led to the growth of interest in the study of the properties of confined degenerate qauntum gases. From the technical point of view the fermions (Cataliotti *et al.*, 1998; DeMarco *et al.*, 1999; McAlexander *et al.*, 1995) can be trapped and cooled in much the same way as bosons. Magnetically trapped fermions are the ideal quantum systems for studying the variety of the effects for degenerate Fermi ensembles. In the same way as bosonic quantum systems, the Fermi gas can provide us with complementary information on the properties of Fermi quantum system at the macroscopic level. In view of the successful experiments with the trapping and cooling of weakly interacting fermionic isotopes it is important to theoretically investigate their thermodynamical and statistical properties. Along with the experiments the behavior of noninteracting Fermi gas has been studied in theoretical works for the approximation of a small number of fermions (Schneider and Wallis, 1998) as well as in the semiclassical Thomas–Fermi approximation (Butts and Rokhsar, 1997; Noronha and Toms, 2000). The purpose of this paper is to examine thermodynamical properties of trapped noninteracting Fermi gases in gravitational fields. In this paper we extend the efforts on the analytical study of thermodynamical behavior of Fermi ensembles taking

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into account the interaction of fermions with the gravitational field. We analyze the influence of nonhomogeneous gravitational field on the specific heat and on the chemical potential of ideal trapped quantum Fermi gas. We will also provide analytical computations of the size of atomic cloud and its dynamics.

For the study of the thermal properties of ideal gas of trapped fermions we will start with the computation of the single particle energy. The single particle Hamiltonian is the sum of kinetic and potential energies

$$H = \frac{\vec{p}^2}{2m} + V_{\text{ext}}, \tag{1}$$

where m is the mass of the particle, and the potential energy is written as the sum of trap potential and gravitational contribution: $V_{\text{ext}} = V_{\text{trap}} + V_{\text{g}}$. The contribution from the harmonic trap to the potential energy has the form

$$V_{\text{trap}} = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right), \tag{2}$$

and the gravitational contribution is $V_{\text{g}} = m\Phi(\vec{X}_0)$. To take into account the interaction with gravitational field, we expand the gravitational potential $\Phi(\vec{X}_0)$ about the center of the magnetic trap with coordinates $\vec{X}_0 = \{x, y, z\}$, and write it in the form

$$\Phi = \Phi_0(\vec{X}_0) - g_i(\vec{X}_0)x^i + \frac{1}{2}\Gamma_{ij}(\vec{X}_0)x^i x^j, \tag{3}$$

where $g_i = -\partial\Phi(\vec{X}_0)/\partial x^i$ is gravitational acceleration and $\Gamma_{ij}(\vec{X}_0) = \partial^2\Phi(\vec{X}_0)/\partial x^i \partial x^j$ are components of gravity gradient tensor. One can introduce a local coordinate system in such a way that the components of gravitational acceleration will be defined as $g_x = g_y = 0$, and $g_z = -|\vec{g}|$, then the trap potential will be

$$V_{\text{trap}} = \frac{m\omega^2}{2}\left(\gamma_x^2(\Gamma)x^2 + \gamma_y^2(\Gamma)y^2 + \gamma_z^2(\Gamma)z^2\right) + mgz + m\Phi_0, \tag{4}$$

where we keep only diagonal (leading) components of gravity gradient tensor for the selected coordinate basis of the trap. The coefficients γ of this equation

$$\gamma_x^2(\Gamma) = \left(1 + \frac{\Gamma_{xx}}{\omega_x^2}\right), \quad \gamma_y^2(\Gamma) = \lambda^2\left(1 + \frac{\Gamma_{yy}}{\omega_y^2}\right), \quad \gamma_z^2(\Gamma) = \lambda'^2\left(1 + \frac{\Gamma_{zz}}{\omega_z^2}\right), \tag{5}$$

depend on diagonal components of gravity gradient tensor and asymmetry parameters of the trap λ, λ'. The angular frequencies are defined here as $\omega_x = \omega$, $\omega_y = \lambda\omega$, and $\omega_z = \lambda'\omega$. Combining (1) and (4), one can finally write the equation for the Hamiltonian (1) as

$$H = \frac{\vec{p}^2}{2m} + \frac{m\omega^2}{2}\left(\gamma_x^2(\Gamma)x^2 + \gamma_y^2(\Gamma)y^2 + \gamma_z^2(\Gamma)z^2\right) + mgz + m\Phi_0. \tag{6}$$

The total single particle energy is found from the solution of time independent Schrödinger equation with Hamiltonian (6) and is written as the sum of two contributions

$$E_{n_x,n_y,n_z}(g,\Gamma) = E_{n_x,n_y,n_z}(\Gamma) + E'(g,\Gamma). \tag{7}$$

The first contribution is

$$E_{n_x,n_y,n_z}(\Gamma) = \hbar\omega[\gamma_x(\Gamma)n_x + \gamma_y(\Gamma)n_y + \gamma_z(\Gamma)n_z], \tag{8}$$

where n_x, n_y, and n_z are positive integers. The second one is

$$E'(g,\Gamma) = \frac{\hbar\omega}{2}[\gamma_x(\Gamma) + \gamma_y(\Gamma) + \gamma_z(\Gamma)] + m\Phi_0 - D^2(g,\Gamma), \tag{9}$$

where $D^2(g,\Gamma) = g^2 m/2(\omega\gamma_z(\Gamma))^2$. The total number of fermions in the trap N and their internal energy U can be computed from the equations

$$N = \sum_{n_x,n_y,n_z=0}^{\infty} (z^{-1}e^{\beta E_{n_x,n_y,n_z}} + 1)^{-1} \tag{10}$$

and

$$U = \sum_{n_x,n_y,n_z=0}^{\infty} E_{n_x,n_y,n_z}(z^{-1}e^{\beta E_{n_x,n_y,n_z}} + 1)^{-1}, \tag{11}$$

where $\beta = 1/T$ is the inverse temperature and z is fugacity. The fugacity $z = \exp(\beta\mu)$ in these equations is defined by the chemical potential $\mu = \mu' - E'(g,\Gamma)$ and absorbs $E'(g,\Gamma)$ contribution.

2. CHEMICAL POTENTIAL, INTERNAL ENERGY AND SPECIFIC HEAT

For the computation of the number of particles and internal energy of the system, we will rewrite the triple sum in Eqs. (10) and (11) in the form of the integral over the single particle energy. To perform this transformation, we represent Eqs. (10) and (11) as

$$N = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \int_0^\infty d\xi_3 (z^{-1}e^{\beta\epsilon} + 1)^{-1} \tag{12}$$

and

$$U = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \int_0^\infty d\xi_3\, \epsilon(z^{-1}\, e^{\beta\epsilon} + 1)^{-1}, \tag{13}$$

where $\epsilon = \hbar\omega[\gamma_x(\Gamma)\xi_1 + \gamma_y(\Gamma)\xi_2 + \gamma_z(\Gamma)\xi_3]$ is the function of new variables ξ_1, ξ_2, and ξ_3. After reducing the number of integrations, we get

$$N = \int_0^\infty \frac{1}{z^{-1}\, e^{\beta\epsilon} + 1} D(\epsilon)\, d\epsilon \tag{14}$$

and

$$U = \int_0^\infty \frac{\epsilon}{z^{-1}\, e^{\beta\epsilon} + 1} D(\epsilon)\, d\epsilon, \tag{15}$$

where the function $D(\epsilon)$ defines the density of states

$$D(\epsilon) = \frac{\epsilon^2}{2\gamma_x(\Gamma)\gamma_y(\Gamma)\gamma_z(\Gamma)\,(\hbar\omega)^3}. \tag{16}$$

To obtain the equation for the number of particles, we will perform the integration introducing the new variables $x = \beta\epsilon$ and $\nu = \ln z$, and using Eq. (35). The result will be

$$\begin{aligned} N &= \frac{1}{2\gamma_x(\Gamma)\gamma_y(\Gamma)\gamma_z(\Gamma)(\hbar\omega)^3}\left(\frac{T}{\hbar\omega}\right)^3 \int_0^\infty \frac{x^2\, dx}{e^{x-\nu} + 1} \\ &= \frac{(T\nu)^3}{6\gamma_x(\Gamma)\gamma_y(\Gamma)\gamma_z(\Gamma)(\hbar\omega)^3}\left(1 + \frac{\pi^2}{\nu^2} + \cdots\right). \end{aligned} \tag{17}$$

From Eq. (17) we find the expansion for the chemical potential at low temperature limit

$$\mu(N, T, \Gamma) = \tilde{E}_{\mathrm{F}}(N, \Gamma)\left[1 - \frac{\pi^2}{3}\left(\frac{T}{\tilde{E}_{\mathrm{F}}(N, \Gamma)}\right)\right], \tag{18}$$

where

$$\tilde{E}_{\mathrm{F}}(N, \Gamma) = \hbar\omega[6N\gamma_x(\Gamma)\gamma_y(\Gamma)\gamma_z(\Gamma)]^{1/3} \tag{19}$$

is the Fermi energy. Degeneracy temperature $T_{\mathrm{F}} = \tilde{E}_{\mathrm{F}}(N, \Gamma)$ depends on gravitational contributions for a fixed number of fermions and trap parameters. For the trap with axial symmetry the ratio $\alpha^{-1}\delta T_{\mathrm{F}}/T_{\mathrm{F}}$, where δT_{F} and α are defined as $\delta T_{\mathrm{F}} = T_{\mathrm{F}}(g, \Gamma) - T_{\mathrm{F}}$ and $\alpha = \Gamma_{zz}/6\omega^2$, is a monotonic function of trap parameter λ' for the fixed components of gravity gradient tensor (Fig. 1). For the spherically symmetric trap ($\lambda' = 1$), the Fermi energy of trapped Fermi gas at gravitational field is of the order of $o(\Gamma_{zz}^2/\omega^4)$ and one can assume that the Fermi temperature T_{F} is not changed.

Let us obtain the equation for the chemical potential at a high temperature limit. For this purpose one can expand the integrand in (14) as a series of powers $z\, e^{-\beta\epsilon}$ in assumption that $z \ll 1$ and write the result as

$$N = \int_0^\infty z\, e^{-\beta\epsilon}(1 - z\, e^{-\beta\epsilon} + \cdots)\; D(\epsilon)\, d\epsilon. \tag{20}$$

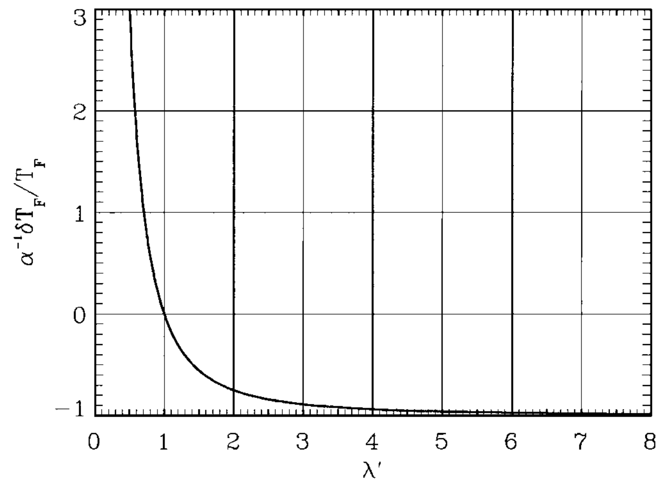

Fig. 1. The ratio $\alpha^{-1}\delta T_{\mathrm{F}}/T_{\mathrm{F}}$ as a function of parameter λ' for the trap with axial symmetry.

Keeping only the first term, we find the fugacity of the Boltzmann gas

$$z_{\mathrm{Bol}}(N, T, \Gamma) = \frac{1}{6}\left(\frac{\tilde{E}_{\mathrm{F}}(N, \Gamma)}{T}\right)^3. \tag{21}$$

The chemical potential is found easily from the last equation

$$\mu_{\mathrm{Bol}}(N, T, \Gamma) = -T \ln\left[6\left(\frac{T}{\tilde{E}_{\mathrm{F}}(N, \Gamma)}\right)^3\right]. \tag{22}$$

The internal energy for the finite Fermi system in a magnetic trap is found from Eqs. (15) and (35). The resulting equation for the internal energy of Fermi gas will be

$$\begin{aligned} U &= \frac{T^4}{2\gamma_x(\Gamma)\gamma_y(\Gamma)\gamma_z(\Gamma)(\hbar\omega)^3}\int_0^\infty \frac{x^3\,dx}{e^{x-\nu}+1} \\ &= \frac{(T\nu)^4}{8\gamma_x(\Gamma)\gamma_y(\Gamma)\gamma_z(\Gamma)(\hbar\omega)^3}\left(1 + 2\frac{\pi^2}{\nu^2} + \cdots\right). \end{aligned} \tag{23}$$

Using Eqs. (17) and (23), we get the asymptotic expansion for the internal energy in the form

$$U(N, T, \Gamma) = \frac{3}{4}N\tilde{E}_{\mathrm{F}}(N, \Gamma)\left[1 + \frac{2\pi^2}{3}\left(\frac{T}{\tilde{E}_{\mathrm{F}}(N, \Gamma)}\right)^2 + \cdots\right]. \tag{24}$$

The specific heat of the trapped fermion in external gravitational field is computed as a derivative of the internal energy $C = \partial U/\partial T$. The resulting equation for a

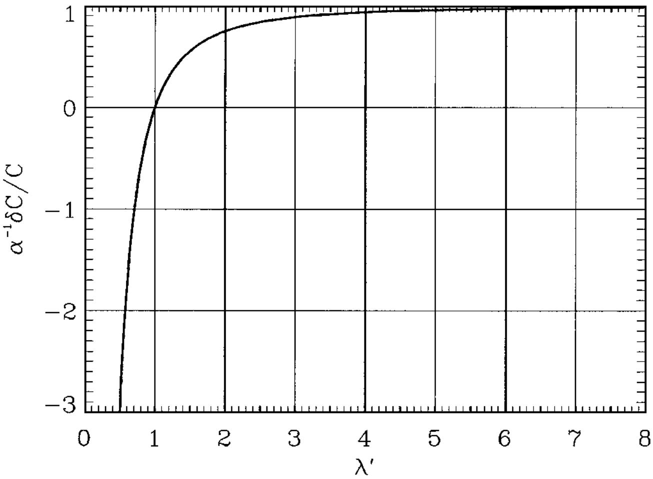

Fig. 2. The ratio $\delta C/C$ as a function of the parameter λ' for the trap with axial symmetry.

specific heat will be

$$\frac{C}{N} = \frac{\pi^2 T}{\tilde{E}_{\mathrm{F}}(N,\Gamma)}. \tag{25}$$

In a high temperature limit we get well-known relation $C/N = 3$. The dependence $\alpha^{-1}\delta C/C$ on λ' for the trap with axial symmetry is given by a plot of Fig. 2. As it has been pointed out, the parameter α depends on component Γ_{zz} of gravity gradient and radial frequency ω. The shift of the specific heat δC is defined as the difference $\delta C = C(g,\Gamma) - C$. For the spherically symmetric trap ($\lambda = \lambda' = 1$), the specific heat of the trapped Fermi gas in gravitational field is of the order of $o(\Gamma_{zz}^2/\omega^4)$.

3. TRAPPED FERMIONS IN SEMICLASSICAL REGIME

In many cases it is helpful to use semiclassical theory to overcome mathematical difficulties in the description of collective phenomena. The semiclassical approach allows us to obtain the spatial distribution for a large number of trapped fermions in external gravitational field. To perform the necessary computations, let us assume that the Fermi ensemble be in equilibrium in varying external potential $\prod(\vec{r},\Gamma)$ of the form

$$\prod(\vec{r},\Gamma) = \frac{m\omega^2}{2}\left(\gamma_x^2(\Gamma)x^2 + \gamma_y^2(\Gamma)y^2 + \gamma_z^2(\Gamma)z^2\right). \tag{26}$$

If all quantum states in the region at $\vec{r}$ are filled to the Fermi energy, then one can write the equation

$$\frac{\hbar^2 k_{\mathrm{F}}^2(\vec{r})}{2m} + \prod(\vec{r}, \Gamma) = \tilde{E}_{\mathrm{F}}, \tag{27}$$

where Fermi energy $\tilde{E}_{\mathrm{F}}$ is the constant of the system, and the local Fermi wave number $k_{\mathrm{F}}(\vec{r})$ is related to fermion density $n(\vec{r})$ according to the equation $k_{\mathrm{F}}^3(\vec{r}) = 6\pi^2 n(\vec{r})$. The equation for the density of particles is found from (27) and has the following form

$$n(\vec{r}, \Gamma) = \frac{1}{6\pi^2}\left[\frac{2m\tilde{E}_{\mathrm{F}}}{\hbar^2}\left(1 - \frac{\tilde{\rho}^2(\vec{r}, \Gamma)}{\tilde{R}_{\mathrm{F}}^2}\right)\right]^{3/2}, \tag{28}$$

where $\tilde{\rho}(\vec{r}, \Gamma) = [\gamma_x^2(\Gamma)x^2 + \gamma_y^2(\Gamma)y^2 + \gamma_z^2(\Gamma)z^2]^{1/2}$ is the effective distance from the center of the ellipsoid to the point $\vec{r}$ with coordinates (x, y, z), and $\tilde{R}_{\mathrm{F}} = [2\tilde{E}_{\mathrm{F}}/m\omega^2]^{1/2}$ defines the axis of the ellipsoid. The ellipsoid axes depend on the components of gravity gradient tensor, and they are defined as $R_i(\Gamma) = 2\tilde{R}_{\mathrm{F}}\gamma_i^{-1}(\Gamma)$, where the index i runs through $i = x, y, z$. As follows from Eqs. (28), the density at the center of the cloud is $n(0) = (2m\tilde{E}_{\mathrm{F}})^{3/2}/6\pi^2\hbar^3$. For the estimation of the fermion density and the size of the ensemble, one can use Eq. (19) for Fermi energy $\tilde{E}_{\mathrm{F}}(N, \Gamma)$. As one can see, the size of the ensemble and the central density depend on the total number of fermions N and the components of gravity gradient tensor. For the trap with axial symmetry the resulting equation for the central density of fermions depends on the third component Γ_{zz}, radial frequency ω, and asymmetry parameter λ':

$$n(0) = \frac{1}{\pi^2}\left(\frac{N\lambda'}{6}\right)^{1/2}\left(\frac{2m\omega}{\hbar}\right)^{3/2}\left[1 - \frac{3\alpha}{2}\left(1 - \frac{1}{\lambda'^2}\right)\right]. \tag{29}$$

As it has been shown, the properties of trapped Fermi gas depend on the characteristics of gravitational field. The expressions for the chemical potential, Fermi energy, internal energy, and the specific heat depend on the diagonal components of gravity gradient tensor.

In the conclusion it could be important to point out two features in the computation of thermodynamical characteristics of trapped fermions. First, the equations for the chemical potential, Fermi energy, total energy, and the specific heat of noninteracting ensemble of degenerate Fermi gas include the interaction with gravitational field by the product $\gamma_x\gamma_y\gamma_z$. Second, the diagonal components of gravity gradient tensor Γ_{ij} are not independent.[2] Gravity gradient tensor has a zero trace, therefore one can introduce only two independent diagonal components. The

[2] The gravity gradient tensor is symmetric $\Gamma_{ij} = \Gamma_{ji}$. Its trace is related to the local mass density ρ by the Poisson's equation $\sum_i \Gamma_{ii} = 4\pi G\rho$, where G is gravitational constant.

harmonic trap also has two independent parameters λ and λ′, which give us the way to sellect the number of independent components of gravity gradient tensor. The contribution of each diagonal component can be estimated independently by an appropriate selection of the parameters of the trap. For the different values of the parameters λ′ and λ′, the product $\gamma_x\gamma_y\gamma_z$ can be written as

$$\gamma_x\gamma_y\gamma_z = \begin{cases} \lambda^2 + (\Gamma_{xx}/2\omega^2)(\lambda^2 - 1) & \lambda = \lambda' \neq 1 \\ \lambda + (\Gamma_{yy}/2\omega^2)(\lambda^{-1} - \lambda) & \lambda \neq 1,\ \lambda' = 1 \\ \lambda' + (\Gamma_{zz}/2\omega^2)(\lambda'^{-1} - \lambda') & \lambda = 1,\ \lambda' \neq 1 \end{cases} \tag{30}$$

and $\gamma_x\gamma_y\gamma_z \approx 1 + o(\Gamma_{ii}\Gamma_{jj}/\omega^4)$ for the trap with axial symmetry ($\lambda = \lambda' = 1$). The results clearly show that for parameters λ and λ′, which are defined in (30), the chemical potential, the total energy, and the specific heat will depend on only one component of gravity gradient tensor. Specifying $\lambda = \lambda' \neq 1$ we obtain that these values depend only on the first component of gravity gradient Γ_{xx}. For $\lambda \neq 1$, $\lambda' = 1$ they will depend on the second component Γ_{yy}, and for $\lambda = 1$, $\lambda' \neq 1$, we find that the chemical potential, the internal energy, and the specific heat depend only on the third component Γ_{zz}. For given Γ_{zz}, the ratios $\alpha^{-1}\delta T_F/T_F$ and $\alpha^{-1}\delta C/C$ for the Fermi temperature and the specific heat are monotonic functions of the parameter λ′ of axially symmetric trap. These functions are shown in Figs. 1 and 2. The shape of the cloud of completly degenerate trapped fermions (gas at a temperature of absolute zero) is defined by Eqs. (28), and has the shape of an ellipsoid.[3]

4. APPENDIX

The low temperature approximation for the number of particles and internal energy can be obtained by the partial integration of the integral

$$\begin{aligned} \int_0^\infty \frac{x^n\,dx}{e^{x-\nu}+1} &= \frac{1}{(n+1)}\int_0^\infty \frac{x^{n+1}\,e^{x-\nu}\,dx}{(e^{x-\nu}+1)^2} \\ &= \frac{1}{(n+1)}\int_{-\nu}^\infty \frac{(t+\nu)^{n+1}\,e^t\,dt}{(e^t+1)^2} \\ &= \frac{1}{(n+1)}\left[\int_{-\infty}^\infty \frac{(t+\nu)^{n+1}\,e^t\,dt}{(e^t+1)^2} - \int_{-\infty}^{-\nu} \frac{(t+\nu)^{n+1}\,e^t\,dt}{(e^t+1)^2}\right]. \end{aligned} \tag{31}$$

[3] The source of gravitational field is not specified in our calculations. For the Earth gravity, for example, the diagonal components of gravity gradient tensor are $\Gamma_E = (GM_E/R^3)\mathrm{diag}\,(-2, 1, 1)$ where $\Gamma_E = (GM_E/R^3) = 1.5 \times 10^3 E$ö.

The second integral in (31) is of the order $o(e^{-\nu})$, and the first one can be found by using a Taylor expansion:

$$\begin{aligned}
&\frac{1}{(n+1)}\int_{-\infty}^{\infty}\frac{(t+\nu)^{n+1}\,e^{t}\,dt}{(e^{t}+1)^{2}}\\
&\quad=\frac{1}{(n+1)}\int_{-\infty}^{\infty}\frac{e^{t}\,dt}{(e^{t}+1)^{2}}\times\left(\nu^{n+1}+\frac{(n+1)}{1!}\nu^{n}t+\frac{n(n+1)}{2!}\nu^{n-1}t^{2}+\cdots\right)\\
&\quad=\frac{1}{(n+1)}\nu^{n+1}I_{0}+\nu^{n}I_{1}+\frac{n}{2}\nu^{n-1}I_{2}+\cdots
\end{aligned}\tag{32}$$

The first integral in this equation is trivial

$$I_0=\int_{-\infty}^{\infty}\frac{e^{t}\,dt}{(e^{t}+1)^{2}}=1. \tag{33}$$

All the integrals for odd n are zeros, and the integrals with even n are given by the equation

$$I_n=\int_{-\infty}^{\infty}\frac{t^{n}\,e^{t}\,dt}{(e^{t}+1)^{2}}=(n-1)!(2n)\,(1-2^{1-n})\zeta(n), \tag{34}$$

where $\zeta(n)$ is the Riemann zeta function. The result will be written in the form

$$\int_{0}^{\infty}\frac{x^{n}\,dx}{e^{x-\nu}+1}=\frac{\nu^{n+1}}{(n+1)}\left[1+\frac{n(n+1)}{6}\frac{\pi^{2}}{\nu^{2}}+\cdots\right], \tag{35}$$

where the expression $I_2=\pi^2/3$ was used.

ACKNOWLEDGMENTS

This research was performed at the Jet Propulsion Laboratory, California Institute of Technology. The author gratefully acknowledges the support of NRO-DII program.

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